THE FIELD OF A SPHERICAL WAVE AT HIGH FREQUENCIES DIFFRACTED BY A SPHERE

by

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1. *Introduction.*

Recently several authors [1-8] have given an asymptotic solution with respect to frequency of the reduced wave equation.

D.S. Jones [1, 2] derived the solution by means of asymptotic expansion of the integral equation which is obtained on treating some simple problems. This method is not easy to handle in the three dimensional case of diffraction around arbitrary convex shapes.

A very useful method is developed by J.B. Keller [3, 4]. This theory employs rays as the basic concept and in this way some diffraction problems are solved by J.B. Keller, R.M. Lewis and B.D. Seckler [5]. However in this theory always some results, which are obtained by other expansion, are used.

A more general method is given by R.N. Buchal and J.B. Keller [6] by means of the application of the known boundary layer expansion theory. In this way expansions are obtained in the case of two dimensional caustics and diffraction of a wave by an aperture in a thin screen.

Boundary layer expansions are also used by V.A. Fock [7] in the two dimensional diffraction of waves by an arbitrary body. E. Zauderer [8] treated the general three dimensional case of diffraction by an arbitrary shape, however, this method does not give all the results without using some expansions which are obtained geometrically by J.B. Keller [3] who uses some known exact expansions. In particular E. Zauderer employs the exponential behaviour in the far field of the shadow region.

In this article we will develop the asymptotic expansion in the shadow region of a body of special shape without using results obtained from other expansions. This method is directly to be used in more general problems.

The problem of diffraction of a time harmonic spherical wave in the shadow region of a sphere will be solved by means of boundary layer expansions. We introduce "ray coordinates". In an homogeneous medium the rays are straight lines. On the sphere creeping rays occur which are geodesic lines starting in the points where the incident rays are tangent to the sphere and which have the direction of the incident rays at these points. The rays in the shadow region are straight lines generated by the creeping rays tangent to the sphere. The latter rays will be used to introduce new coordinates and can be used in general cases of diffraction of general incident waves by convex bodies.

In this article the method is carried out in the case of high frequency scattering of a spherical wave by a sphere. We consider only the first term of each asymptotic series supposing it exists.

In this article no further justification of the asymptotic expansions is given. This will be done in a thesis which will be published soon.

2. *Formulation of the problem.*

We consider the field of a spherical source at the point $P(-\rho, 0, 0)$ diffracted by a sphere and solve this problem in the case of high frequencies.

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The scalar function $\tilde{\varphi}$ has to be a solution of the reduced wave equation,

$$\Delta\tilde{\varphi} + k^2\tilde{\varphi} = 0 \tag{1}$$

with large values of k.

This function $\tilde{\varphi}$ must vanish on a sphere with radius one and with the center in the origin.

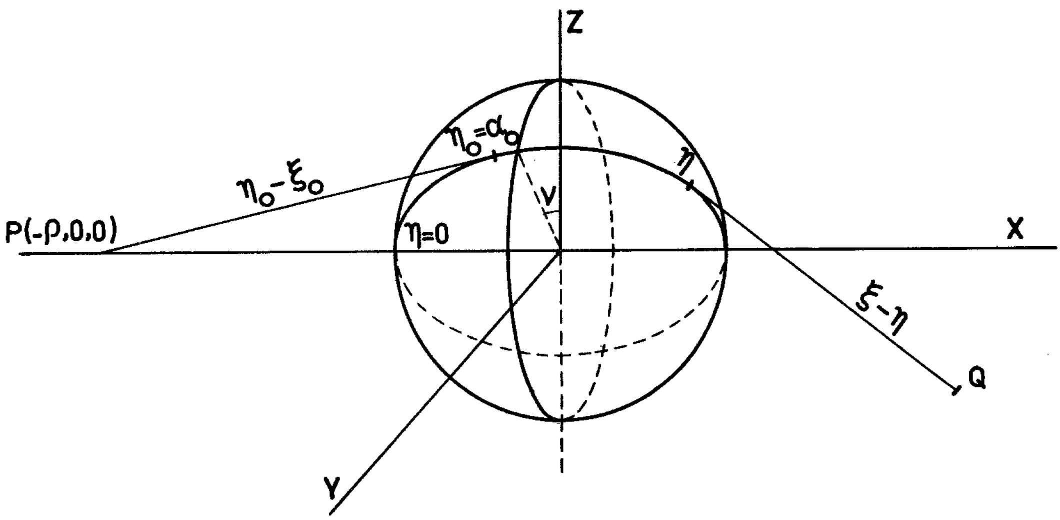

Fig. 1.

The problem is axially symmetric and we introduce the new coordinates ξ, η and θ as defined in figure 1.

$\xi - \eta$ is the arclength along the ray

η is the arclength on the geodesic

θ is the polar angle.

We define ξ_0 in such a way that the arclength PQ along the ray is equal to $\xi - \xi_0$. This coordinate system is not one valued.

A point Q(x, y, z) is determined by

$$\left.\begin{aligned} \xi_{m\ell} &= \xi_\ell + 2\pi m \\ \eta_{m\ell} &= \eta_\ell + 2\pi m \end{aligned}\right\} \quad m = 0, 1, 2, \ldots$$
$$\theta_\ell = \theta + \ell\pi \qquad \ell = 0, 1 \tag{2'}$$

The point Q is reached by rays generated from two different points on the sphere, however the arclength of the creeping rays generating from one of these points differs $2\pi m$ and therefore the amplitudes are different.

Hence the solution $\tilde{\varphi}(x, y, z)$ is a superposition of the corresponding solutions

$$\tilde{\varphi}(x, y, z) = \sum_{m=0}^{\infty} \tilde{\varphi}(\xi_{m0}, \eta_{m0}, \theta_0) + \sum_{m=0}^{\infty} \tilde{\varphi}(\xi_{m1}, \eta_{m1}, \theta_1) \tag{2}$$

The new coordinates are suitable to give us the solution in the shadow region and near the shadow boundary. In the lit region we find the total field by adding the geometrical field to the diffracted field.

In this article we only consider the diffracted field in the shadow region.

The function $\tilde{\varphi}$ is independent of θ, so in the new coordinates we get the differential equation

$$\Delta\tilde{\varphi} + k^2\tilde{\varphi} = \frac{1}{(\xi-\eta)f(\xi,\eta)}\left[\frac{\partial}{\partial\xi}(\xi-\eta)f(\xi,\eta)\frac{\partial\tilde{\varphi}}{\partial\xi} + \frac{\partial}{\partial\eta}\frac{f(\xi,\eta)}{\xi-\eta}\frac{\partial\tilde{\varphi}}{\partial\eta}\right] + k^2\tilde{\varphi} = 0 \quad (3)$$

with $f(\xi,\eta) = \sin\eta + (\xi-\eta)\cos\eta$.

We substitute

$$\tilde{\varphi}(\xi,\eta) = \varphi(\xi,\eta)e^{ik(\xi-\xi_o)} \quad (4)$$

The new differential equation is

$$\frac{1}{\xi-\eta}\frac{\partial}{\partial\xi}(\xi-\eta)\frac{\partial\varphi}{\partial\xi} + \frac{1}{\xi-\eta}\frac{\partial\varphi}{\partial\eta}\frac{1}{\xi-\eta}\frac{\partial\varphi}{\partial\eta} + \frac{\partial\varphi}{\partial\xi}\frac{f_\xi}{f} +$$

$$+ \frac{1}{(\xi-\eta)^2}\frac{\partial\varphi}{\partial\eta}\frac{f_\eta}{f} + ik\left\{2\frac{\partial\varphi}{\partial\xi} + \frac{\varphi}{\xi-\eta} + \varphi\frac{f_\xi}{f}\right\} = 0 \quad (5)$$

with the boundary condition

$$\varphi = 0 \quad \text{if} \quad \xi = \eta$$

and the incident part of φ is equal to the incident wave on the shadow boundary:

$$\tilde{\varphi}_{inc} = \frac{e^{ik(\xi-\xi_o)}}{\xi-\xi_o} \quad \text{if} \quad \eta = \eta_o = \alpha_o. \quad (6)$$

We try to find a solution with these boundary conditions at finite distance from the sphere.

3. *First approach to the approximation at finite distance from the sphere.*

We approximate the value of φ for large values of k. We suppose equation (5) has a solution of the form [10]

$$\varphi_o(\xi,\eta;k) = e^{ik^s\Psi(\eta)}\sum_{n=0}^{\infty} k^{-n}\varphi_{on}(\xi,\eta) \quad (7)$$

with $s < \frac{1}{2}$.

The function ψ has to be a constant along a ray.

When (7) is inserted into (5) we equate the coefficient of each power of k equal to zero.

The first equation, which gives the first approximation, is

$$2\frac{\partial\varphi_{oo}}{\partial\xi} + \frac{\varphi_{oo}}{\xi-\eta} + \varphi_{oo}\frac{f_\xi}{f} = 0 \quad (8)$$

This is a first order differential equation with the solution

$$\varphi_{oo}(\xi,\eta) = \frac{A(\eta)}{\sqrt{(\xi-\eta)f(\xi,\eta)}} \quad (9)$$

This solution is singular if $\xi-\eta = 0$, that is on the sphere, or if $f(\xi,\eta) = 0$ on the line through the source and the center of the sphere. We have to satisfy a boundary condition at infinity or a matching condition for $\xi-\eta \to 0$

to determine $A(\eta)$, s, and $\Psi(\eta)$. We have no condition at infinity, so we first have to construct a solution for small values of $\xi-\eta$.

4. *Boundary layer expansion near the sphere and the shadow boundary.*

Near the sphere and the shadow boundary ξ and η both tend to α_0, so we define the new variables α and β

$$\left.\begin{aligned} \xi &= \alpha_0 + k^{-\lambda}\alpha \\ \lambda &= \alpha_0 + k^{-\lambda}\beta \end{aligned}\right\} \tag{10}$$

The value of λ has to be determined by the stretching condition. If we put the new variables (10) into equation (5) this yields the new equation

$$\frac{k^{2\lambda}}{\alpha-\beta}\frac{\partial}{\partial\alpha}(\alpha-\beta)\frac{\partial\varphi}{\partial\alpha}+\frac{k^{4\lambda}}{\alpha-\beta}\frac{\partial}{\partial\beta}\frac{1}{\alpha-\beta}\frac{\partial\varphi}{\partial\beta}+k^{\lambda}\frac{\partial\varphi}{\partial\alpha}\frac{f_\xi}{f}+$$
$$+\frac{k^{3\lambda}}{\alpha-\beta}\frac{\partial\varphi}{\partial\beta}\frac{f_\eta}{f}+ik\left\{2k^{\lambda}\frac{\partial\varphi}{\partial\alpha}+k^{\lambda}\frac{\varphi}{\alpha-\beta}+\varphi\frac{f_\xi}{f}\right\}=0 \tag{11}$$

We suppose $f(\xi,\eta) = 0(1)$ in this region, then the stretching condition requires that $1+\lambda = 4\lambda$ from which it follows that

$$\lambda = \frac{1}{3} \tag{12}$$

Now we assume φ has the expansion

$$\varphi_1(\alpha,\beta;k) = k^r \sum_{n=0}^{\infty} k^{-n/3}\varphi_{1n}(\alpha,\beta) \tag{13}$$

and if we put this in equation (11) we put each coefficient of each power of k equal to zero. The first term of this expansion is a solution of the equation

$$\frac{1}{\alpha-\beta}\frac{\partial}{\partial\beta}\frac{1}{\alpha-\beta}\frac{\partial\varphi_{10}}{\partial\beta}+2i\frac{\partial\varphi_{10}}{\partial\alpha}+i\frac{\varphi_{10}}{\alpha-\beta}=0 \tag{14}$$

with boundary condition

$$\varphi_{10} = 0 \quad \text{if} \quad \alpha=\beta.$$

With the substitution

$$\varphi_{10} = u\cdot\exp\left\{\frac{-i(\alpha-\beta)^3}{3}\right\}$$

and the new variables

$$\left.\begin{aligned} x &= 2^{-1/3}\alpha \\ y &= 2^{-2/3}(\alpha-\beta)^2 \end{aligned}\right\} \tag{15}$$

we get the parabolic equation

$$\frac{\partial^2 u}{\partial y^2}+i\frac{\partial u}{\partial x}+yu=0 \tag{16}$$

Fock [7] gives the solution of this equation for $x > 0$ and $y_o > y$

$$u = B \int_c e^{it(x-x_o)} w(t-y_o) \left[v(t-y) - \frac{v(t)}{w(t)} w(t-y)\right] dt \tag{17}$$

In this formula x_o and y_o are x, y coordinates of the source and the contour C encloses the first quadrant of the complex t-plane.
The function w(t) is a solution of the equation

$w''(t) = t\, w(t)$ and is defined by the following Airy function

$$w(t) = e^{\frac{2\pi i}{3}} \sqrt{\frac{\pi}{3}} (-t)^{1/2} H^{(1)}_{1/3} \left\{\frac{2}{3}(-t)^{3/2}\right\} \tag{18}$$

With $H^{(1)}_{1/3}$ is the Hankel function of the first kind and the order 1/3. The function v(t) satisfies the same differential equation and is equal to the imaginary part of w(t) for real t.
As a solution of our problem we take

$$\varphi_{10} = B \exp\left\{-i \frac{2}{3} (y^{3/2} + y_o^{3/2})\right\} \int_c e^{it(x-x_o)} w(t-y_o) \left\{v(t-y) - \frac{v(t)}{w(t)} w(t-y)\right\} dt \tag{19}$$

if $x > 0$ and $y_o > y$

and $\varphi_1 \approx k^r \varphi_{10}$.

We are also able to evaluate a solution of the lit region where $x < 0$, but we have to take an other combination of Airy functions and close the contour in the fourth quadrant. This gives us the possibility to give the solution in the whole glancing area. However, in this article we only consider the shadow region.
We evaluate integral (19) as a sum residues

$$\varphi_1 \approx -2\pi i k^r B \exp\left\{-i \frac{2}{3} (y^{3/2} + y_o^{3/2})\right\} \sum_{s=1}^{\infty} e^{it_s(x-x_o)} \frac{w(t_s - y_o)\, w(t_s - y)}{\{w^1(t_s)\}^2} \tag{20}$$

The poles $t = t_s$ are the zeros of the function w(t). These are points on the line $t = \rho e^{\frac{\pi i}{3}}$ for real positive ρ, [9].
Evaluating (20) for large values of $y = (\frac{k}{2})^{2/3} (\xi - \eta)^2$ and $y_o = (\frac{k}{2})^{2/3} (tg\, \alpha_o)^2$, we get the result

$$\varphi_1 \approx \frac{2\pi B k^{r-1/3} 2^{1/3}}{\sqrt{tg\, \alpha_o (\xi - \eta)}} \sum_{s=1}^{\infty} \frac{\exp\left\{i (\frac{k}{2})^{1/3} t_s (\eta - \alpha_o)\right\}}{\{w^1(t_s)\}^2} \tag{21}$$

We have to match this solution with the solution at finite distance from the sphere. As we see, we have to take instead of (9) a solution of the form,

$$\varphi_o \approx \frac{2\pi D k^p}{\sqrt{(\xi - \eta) \{tg\, \eta + (\xi - \eta)\}}} \sum_{s=1}^{\infty} \frac{\exp\left\{i (\frac{k}{2})^{1/3} t_s (\eta - \alpha_o)\right\}}{\{w^1(t_s)\}^2} \tag{22}$$

which is an asymptotic solution of (5).
If we match these two solutions in the region $\xi - \eta \approx 0$ and $\eta \approx \alpha_o$ we find $p = r - 1/3$ and $D = 2^{1/3} B$ and if we write (22) in the integral form like (19) we find the solution

$$\varphi_o \approx \frac{k^r \; B \sqrt{\operatorname{tg}\alpha_o}}{\sqrt{\operatorname{tg}\eta + (\xi-\eta)}} \exp\left\{-i\,\frac{2}{3}(y^{3/2}+y_o^{3/2}\right).$$

$$\cdot \int_c e^{it(x-x_o)} w(t-y_o)\left\{v(t-y) - \frac{v(t)}{w(t)}\, w(t-y)\right\} dt \tag{23}$$

The constants r and B have to be determined with the condition that on the shadow boundary the incident part of (23) is the same as the incident wave. In the region $\eta \approx \alpha_o$ solution (22) converges very slowly, so we have to evaluate (23) by the method of stationary phase (Fock [7]).

We find the relation

$$\varphi_o \approx \frac{k^{r-1/6}\, B \sqrt{\pi \operatorname{tg}\alpha_o}\; 2^{1/6}\, e^{i\frac{\pi}{4}}}{\operatorname{tg}\alpha_o + \xi-\alpha_o} \equiv \frac{1}{\operatorname{tg}\alpha_o + \xi-\alpha_o} \tag{24}$$

So we find

$$\begin{aligned} r &= 1/6 \\ B &= \frac{2^{-1/6}\, e^{-i\frac{\pi}{4}}}{\sqrt{\pi \operatorname{tg}\alpha_o}} \end{aligned} \tag{25}$$

In this way we constructed a solution at infinity for large values of $\xi-\eta$ with $\xi-\alpha_o$ large enough to have convergence

$$\varphi_o \approx \frac{2\pi\,(\frac{k}{2})^{-1/6}\, e^{-\frac{\pi i}{4}}}{\sqrt{\pi\operatorname{tg}\alpha_o}\,\sqrt{(\xi-\eta)\,\{\operatorname{tg}\eta + (\xi-\eta)\}}} \sum_{s=0}^{\infty} \frac{\exp\left\{i\,(\frac{k}{2})^{1/3}\, t_s(\eta-\alpha_o)\right\}}{\left\{w^1(t_s)\right\}^2} \tag{26}$$

It is easy to show by taking the sum of residues that (23) also holds near the sphere if $\eta \gg \alpha_o$ except in the region where $\operatorname{tg}\eta + (\xi-\eta) \approx 0$.

5. *Boundary layer expansion near the caustic.*

The given expansions are valid in the region $f(\xi,\eta) = 0$ (1). We shall investigate the behaviour of the field near the line $f(\xi,\eta) = 0$. Knowing $f(\xi,\eta)$ is the distance from the observation point to the caustic, we formulate new variables α and β.

$$\left.\begin{aligned} \sin\eta + (\xi-\eta)\cos\eta &= k^{-\mu}\alpha \\ \xi-\eta &= \beta \end{aligned}\right\} \tag{1a}$$

We suppose that the solution has the form

$$\varphi_2(\xi,\eta;k) = \sum_{s=1}^{\infty} u(\xi,\eta;s,k)\,\exp\left\{i\,(\frac{k}{2})^{1/3}\, t_s(\eta-\alpha_o)\right\} \tag{2a}$$

and find for each $u(\xi,\eta;s,k)$ the differential equations in the new variables,

$$\frac{1}{\beta}\,(k^{\mu}\cos\eta\,\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta})\left\{\beta(k^{\mu}\cos\eta\,\frac{\partial u}{\partial\alpha} + \frac{\partial u}{\partial\beta})\right\} +$$

$$+ \frac{1}{\beta}(-i(\frac{k}{2})^{1/3} t_s + k^{\mu}\beta\sin\eta\,\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta})\left\{\frac{1}{\beta}(-i(\frac{k}{2})^{1/3} t_s u + k^{\mu}\beta\sin\eta\,\frac{\partial u}{\partial\alpha} + \frac{\partial u}{\partial\beta})\right\} +$$

$$+\frac{k^{\mu}\cos\eta}{\alpha}\left(k^{\mu}\cos\eta\,\frac{\partial u}{\partial\alpha}+\frac{\partial u}{\partial\beta}\right)+\frac{k^{\mu}\sin\beta}{\alpha\beta}\left\{-i\left(\frac{k}{2}\right)^{1/3}t_s u+k^{\mu}\beta\sin\eta\,\frac{\partial u}{\partial\alpha}+\frac{\partial u}{\partial\beta}\right\}+$$

$$+ik\left\{2\left(k^{\mu}\cos\eta\,\frac{\partial u}{\partial\alpha}+\frac{\partial u}{\partial\beta}\right)+\frac{u}{\beta}+\frac{k^{\mu}\cos\eta}{\alpha}\,u\right\}=0 \tag{3a}$$

The stretching condition requires that $1+\mu = 2\mu$, from which it follows that $\mu = 1$.

In the neighbourhood of the caustic we assume that $u(\alpha,\beta;k,s)$ can be expanded in the following series

$$u(\alpha,\beta;k,s) = k^{t}\sum_{n=0}^{\infty} u_n(\alpha,\beta;s)k^{-n} \tag{4a}$$

If we put this in equation (3a) and equate each power of k equal zero, we get the first approximation

$$\frac{\partial^2 u_o}{\partial\alpha^2}+\frac{1}{\alpha}\frac{\partial u_o}{\partial\alpha}+2i\cos\eta\,\frac{\partial u_o}{\partial\alpha}+i\,\frac{\cos\eta}{\alpha}\,u_o=0$$

with $\cos\eta \approx \dfrac{1}{\sqrt{1+\beta^2}}$.

We find the following solution

$$\varphi_2(\alpha,\beta;k)=k^{t}\sum_{s=1}^{\infty}K(\beta,s)\exp\left\{i\left(\frac{k}{2}\right)^{1/3}t_s(\eta-\alpha_o)\right\}\cdot\exp\left\{-i\alpha\cos\eta\right\}H_o^{(1)}(\alpha\cos\eta) \tag{5a}$$

This solution has the required exponential behaviour for large values of α. The constant $K(\beta,s)$ is determined by a matching condition with the solution near the sphere on the same ray.

We find the constants

$$K(\beta,s)=\frac{2\pi i\,2^{-1/3}}{\sqrt{\mathrm{tg}\,\alpha_o}}\,\frac{\cos\eta}{\sqrt{\xi-\eta}}\,\frac{1}{\{w^{1}(t_s)\}} \tag{6a}$$

and $t = 1/3$.

In this way we found the solution near the caustic, however (5a) is singular on the caustic where $\alpha = 0$. A regular solution will be obtained by adding the solution of the second ray to this solution

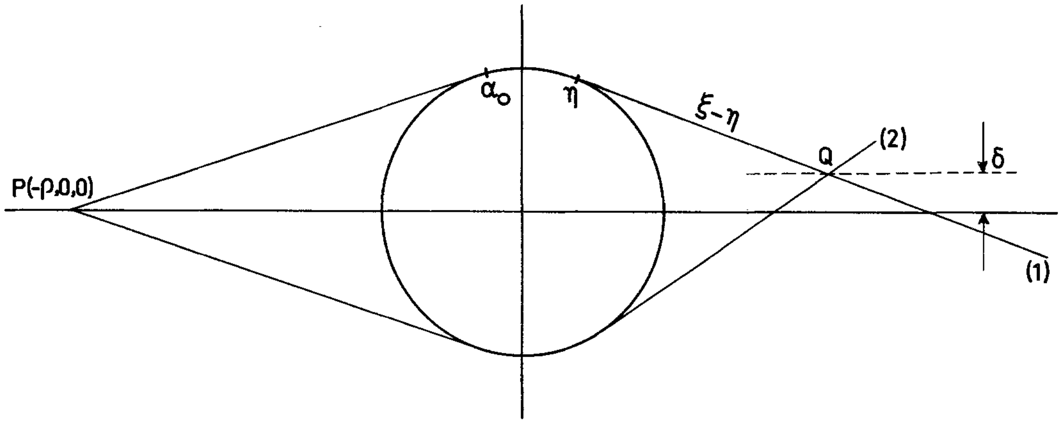

Fig. 2.

The total result of the two rays is finite if $\delta \rightarrow 0$ (with $\delta = \frac{|\alpha|}{k}$).
We get instead of

$$\lim_{\delta \to 0} e^{-i\delta k \cos\eta} H_o^{(1)}(\delta k \cos\eta) \rightarrow \infty$$

the finite limit

$$\lim_{\delta \to 0} \left\{ e^{-i\delta k \cos\eta} H_o^{(1)}(\delta k \cos\eta) + e^{i\delta k \cos\eta} H_o^{(2)}(\delta k \cos\eta) \right\} = 2$$

Until now we investigated the solution of (2) with m = 0, l = 0 or 1. To find the total solution we have to summarize over all the different coordinates.

We consider one side of the polar axis. The result on the other side is the same if we consider axial symmetry.

6. *Final results in the shadow region.*

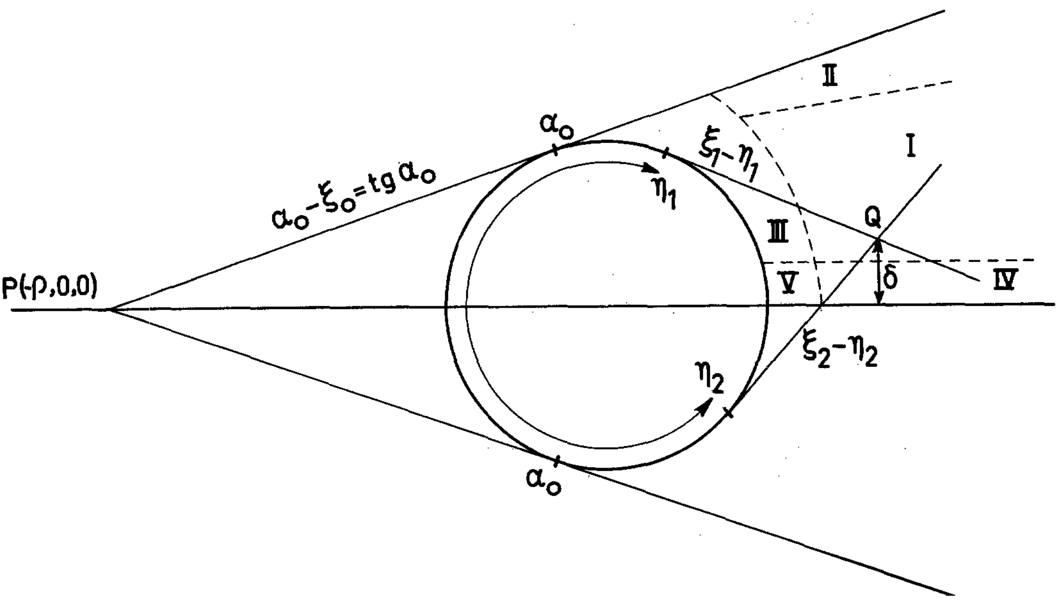

Fig. 3.

In the shadow region we find the following results
a) In region I

$$\eta_1 \gg \alpha_o \qquad \xi_1 \gg \eta_1 \qquad \text{and} \qquad \delta = 0\,(1)$$
$$\eta_2 \gg \alpha_o \qquad \xi_2 \gg \eta_2$$

we find with (2), (4) and (26) the solution

$$\tilde{\varphi}_I(Q) \approx \frac{2\pi \left(\frac{k}{2}\right)^{1/6}}{\sqrt{\pi\, tg\alpha_o}} e^{-\frac{\pi i}{4}} \exp\left\{ik(tg\alpha_o - \alpha_o)\right\}$$

$$\left[e^{ik\xi_1} \sum_{s=0}^{\infty} \frac{\exp\left\{i\left(\frac{k}{2}\right)^{1/3} t_s(\eta_1 - \alpha_o)\right\} \left\{1 - \exp(2k\pi i + 2\left(\frac{k}{2}\right)^{1/3} t_s \pi i\right\}^{-1}}{\left\{w^1(t_s)\right\}^2 \sqrt{(\xi_1 - \eta_1)}\left\{tg\eta_1 + (\xi_1 - \eta_1)\right\}} + \right.$$

$$+ e^{ik\xi_2} \sum_{s=0}^{\infty} \frac{\exp\left\{i(\frac{k}{2})^{1/3} t_s(\eta_2 - \alpha_o)\right\} \left\{1 - \exp(2k\pi i + 2(\frac{k}{2})^{1/3} t_s \pi i)\right\}^{-1}}{\left\{w^1(t_s)\right\}^2 \sqrt{(\xi_2 - \eta_2)\left\{tg\,\eta_2 + (\xi_2 - \eta_2)\right\}}} \Bigg] \quad (7a)$$

b) In region II

$$\eta_1 \approx \alpha_o \quad \text{and} \quad \xi_1 \gg \eta_1$$

it is sufficient to take as an asymptotic solution only the first term of series (2) with $l = 0$ which is asymptotically the incoming part of the wave. So with (23) and (2) we get the solution

$$\tilde{\varphi}_{II}(Q) \approx \frac{(\frac{k}{2})^{1/6} e^{-\frac{\pi i}{4}} \pi^{-\frac{1}{2}} \exp\left\{-i\frac{2}{3}(y_1^{3/2} + y_o^{3/2})\right\}}{\sqrt{tg\,\eta_1 + (\xi_1 - \eta_1)}}$$

$$e^{ik(\xi_1 - \xi_o)} \int_C e^{it(x - x_o)} w(t - y_o) \left\{v(t - y_1) - \frac{v(t)}{w(t)} w(t - y_1)\right\} dt \quad (8a)$$

with $x = (\frac{k}{2})^{1/3}(\xi - \alpha_o)$

$y = (\frac{k}{2})^{2/3}(\xi - \eta)^2$

and the contour C encloses the first quadrant in positive direction. This solution can also be used in the lit region, however, we have to change the Airy function and the contour C.

c) In the neighbourhood of the object (region III), where $\xi \approx \eta$ and $\delta = 0(1)$, we also find a good asymptotic expansion by taking the first term of both series of (2). Formula (23) and (2) give the solution

$$\tilde{\varphi}_{III}(Q) \approx (\frac{k}{2})^{1/6} e^{-\frac{\pi i}{4}} \pi^{-\frac{1}{2}} \exp\left\{-i^{2/3} y_o^{3/2} - ik\xi_o\right\}.$$

$$\sum_{n=1}^{2} \frac{\exp(-i\frac{2}{3} y_n^{3/2} - ik\xi_n) \int_C e^{it(x_n - x_o)} w(t - y_o)\left\{v(t - y_n) - \frac{v(t)}{w(t)} w(t - y_n)\right\} dt}{\sqrt{tg\,\eta_n + (\xi_n - \eta_n)}} \quad (9a)$$

and if $\eta \gg \alpha_o$ we may take the sum of the residues.

d) In region IV near the caustic on which $\delta \approx 0$ we have to make a superposition of solutions like (5a).
In this region $\eta_1 \approx \eta_2 = \eta$ and

$$\xi_1 - \eta_1 \approx \xi_2 - \eta_2 = \xi - \eta \text{ with}$$

$$\alpha = k(\sin\eta + (\xi - \eta)\cos\eta) = k\delta.$$

The result is

$$\tilde{\varphi}_{IV}(Q) \approx \frac{2\pi i(\frac{k}{2})^{1/3}\cos\eta}{\sqrt{tg\,\alpha_o(\xi - \eta)}} \left[\exp(-i\alpha\cos\eta) H_o^{(1)}(\alpha\cos\eta) + \exp(i\alpha\cos\eta) H_o^{(2)}(\alpha\cos\eta)\right].$$

$$\sum_{s=0}^{\infty} \frac{\exp\left\{i(\frac{k}{2})^{1/3} t_s(\eta - \alpha_o)\right\} \exp\left\{ik(tg\,\alpha_o - \alpha_o + \xi)\right\}}{\left\{w^1(t_s)\right\}^2 \left[1 - \exp\left\{2k\pi i + 2(\frac{k}{2})^{1/3} t_s \pi i\right\}\right]} \quad (10a)$$

e) In region V $\xi \approx \eta$ and $\delta = o$.

We find the integral representation by matching $\tilde{\varphi}_{III}$ and $\tilde{\varphi}_{IV}$

$$\tilde{\varphi}_V(Q) \approx (\frac{k}{2})^{2/3} \pi^{-1} \cos\eta \left[\exp(-i\alpha\cos\eta)H_o^{(1)}(\alpha\cos\eta) + \exp(i\alpha\cos\eta\, H_o^{(2)}(\alpha\cos\eta)\right].$$

$$. \exp\left\{-i\frac{2}{3}(y^{3/2} + y_o^{3/2})\right\} e^{ik(\xi-\xi_o)} \int_c e^{it(x-x_o)} w(t-y_o) \left\{v(t-y) - \frac{v(t)}{w(t)} w(t-y)\right\} dt \quad (11a)$$

In this region it is better to develop the integral as a sum of residues. We get a convergent serie because the zeros t_s of w(t) are points of the line $t = \rho e^{\frac{\pi i}{3}}$.

Expansion of $w(t_s - y)$ is not allowed if y is small.

$$\tilde{\varphi}_V(Q) \approx 2i(\frac{k}{2})^{2/3} \cos\eta \left[\exp(-i\alpha\cos\eta)H_o^{(1)}(\alpha\cos\eta) + \exp(i\alpha\cos\eta) H_o^{(2)}(\alpha\cos\eta)\right].$$

$$. \exp\left\{-i\frac{2}{3}(y^{3/2} + y_o^{3/2})\right\} e^{ik(\xi-\xi_o)} \sum_{s=0}^{\infty} e^{it_s(x-x_o)} \frac{w(t_s - y_o)\, w(t_s - y)}{\{w(t_s)\}^2}$$

with $x = (\frac{k}{2})^{1/3} (\xi - \alpha_o)$

$y = (\frac{k}{2})^{2/3} (\xi - \eta)^2$

As we see the method applied in this article gives the asymptotic expansion in the entire shadow region of the sphere. It is also possible to give the solution in the lit region. This gives no more difficulties. The reasoning of this article also holds in the case of incident plane waves and diffraction around bodies of arbitrary shapes.

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